Bazinga! Maths 2019

Math and Physics Club, IIT Bombay

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1 Prelims

1. A student didn't notice a multiplication sign between two 3 digit numbers and wrote it as a single 6 digit number. The resultant number was 7 times more than what it should have been.
   Find the numbers.

2. An algebraic number is defined to be any real number that is a root of a non-zero polynomial (in one variable) with integer coefficients.
   Fill in the blank with the biggest subset A of $S = \{\sin 1°, \cos 1°\}$ such that all elements of A are algebraic. (A may be the empty set.)

3. For $n \in \mathbb{N}$, let a_n denote the number of ordered pairs $(a, b) \in \mathbb{N}^2$ such that:
   $$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$$
   Find the smallest n such that $a_n = 485$.
   Submit your answer by writing n in its prime factorisation or by writing '∞', if no such n exists.

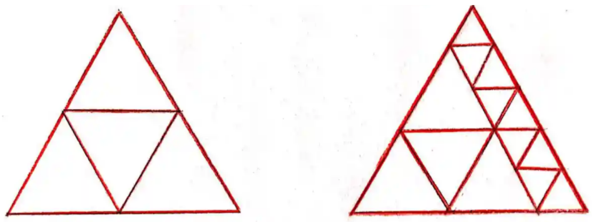
4. If $a_1, a_2, \cdots, a_n$ are n distinct odd natural numbers, not divisible by an prime greater than 5, then find the minimum value of the constant L such that:
   $$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \leq L \qquad \forall n \in \mathbb{N}$$

5. Find the last 8 digits in the binary expansion of 27^{1986}.

6. Let $S = \left\{ z \in \mathbb{C} \;\middle|\; \frac{\sqrt{2}}{2} \leq \mathrm{Re}(z) \leq \frac{\sqrt{3}}{2} \right\}$.
   Find the smallest value of $p \in \mathbb{N}$ such that for all integers $n \geq p$, there exists $z \in S$ such that $z^n = 1$.

7. Let $\lfloor . \rfloor$ denote the greatest integer function.
   Find the value of $\lfloor P \rfloor$ where P is given by:
   $$P := \frac{\sum_{n=1}^{99} \sqrt{10 + \sqrt{n}}}{\sum_{n=1}^{99} \sqrt{10 - \sqrt{n}}}$$

8. An equilateral triangle is divided into smaller equilateral triangles.
   The figure shows that it is possible to divide it into 4 and 13 equilateral triangles. What are the integer values of n, where $n > 1$, for which it is possible to divide the triangle into n smaller equilateral triangles?



9. Find all the values of a in $\{2, 3, \ldots, 999\}$ for which $a^2 - a$ is divisible by 1000.

10. All the terms in the sequence (a_n) are positive real numbers and the sequence (a_n) satisfies the equation below for all positive integers n :

$$\sum_{k=1}^{n} a_k = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right).$$

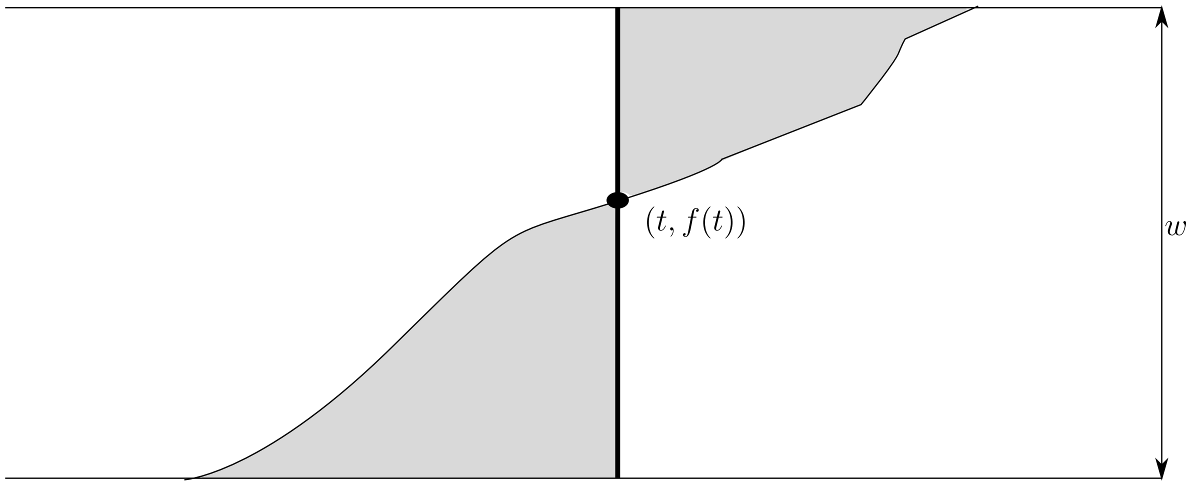
Find the value of

$$\sum_{k=1}^{100} a_k.$$

2 Rapid Fire

Q 2.1. Find the sum of all natural numbers a such that $a^2 - 16a + 67$ is a perfect square.

Q 2.2. The graph of a monotonically increasing continuous function is cut off with two horizontal lines. Find a point on the graph between intersections such that the sum of the two areas bounded by the lines, the graph and the vertical line through the point is minimum. See figure.



Minimise the area

Q 2.3. Find the remainder left when $(2500)(98!)^2$ is divided by 101.

Q 2.4. Find the following limit:

$$\lim_{x \to 0} \frac{\sin \tan \arcsin x - \tan \sin \arctan x}{\tan \arcsin \arctan x - \sin \arctan \arcsin x}$$

Q 2.5. Find the minimum value of $|\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x|$ for real x.

Q 2.6. Let's start with an ordered pair of integers, (a, b). We start a process, where we keep applying the following transformation T :

$$T(a, b) = \begin{cases} (2a, b - a) & \text{if } a < b \\ (a - b, 2b) & \text{otherwise} \end{cases}$$

We say that the process terminates for a given pair (a, b) if there exists an $n \in \mathbb{N}$ such that $T^n(a, b) = T^{n+1}(a, b)$,
where T^n denotes the composition of T with itself n times.
For which of the given pairs does the process terminate?

1. (24, 101)
2. (97, 31)
3. (34, 56)

4. (19, 79)

Q 2.7. $\sqrt{\sqrt[3]{5}-\sqrt[3]{4}} \times 3 = \sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}$ where a, b, c are positive integers. What is the value of $a+b+c$?

Q 2.8. Let S be a finite non-empty set of real positive numbers.
If S contains at least n elements, it is guaranteed that there are elements $x, y \in S$ such that

$$0 < \frac{y-x}{1+xy} < \sqrt{2}-1$$

What is the smallest such n?

Q 2.9. Starting with the vertices $P_1 = (0,1)$, $P_2 = (1,1)$ $P_3 = (1,0)$, $P_4 = (0,0)$ of a square, we construct further points as follows:

P_n is the midpoint of the line segment $\overline{P_{n-4}P_{n-3}}$ for $n \geq 5$.

The spiral approaches a point $P = \lim_{n\to\infty} P_n$. Find P.

Q 2.10. Arrange the integers $1, 2, 3, \cdots, 10$ in some order, and get the sequence $a_1, a_2, a_3, \cdots, a_{10}$.
The sequence satisfies that the unit digit of $a_n + n$ are all different for $n = 1, 2, 3, \cdots, 10$.
How many such arrangements are possible?

Q 2.11. In how many ways can we transform $f(x) = x^2+4x+3$ into $g(x) = x^2+10x+9$ by a sequence of transformations of the form

$$f(x) \mapsto x^2 f\left(\frac{1}{x}+1\right)$$

or

$$f(x) \mapsto (x-1)^2 f\left(\frac{1}{x-1}\right)?$$

(You may answer with infinite as well.)

Q 2.12. You are given a rope of length 1. You pick a real number x randomly from $(0, 1)$ with uniform distribution. You then cut as many segments of length x as possible. In other words, you cut the length nx where n is the largest integer such that $nx \leq 1$.
What is the expected length of the rope remaining?

Q 2.13. For an arbitrary n, let $g(n)$ be the GCD of $2n+1$ and $2n^2+7n+17$. What is the largest positive integer that can be obtained as the value of $g(n)$? If $g(n)$ can be arbitrarily large, state so explicitly.

Q 2.14. Let $x \in \mathbb{C}$ be such that

$$x + x^{-1} = \frac{\sqrt{5}+1}{2}.$$

What is the value of $x^{2019} + x^{-2019}$?

Q 2.15. 100 numbers $1, 1/2, 1/3, \cdots 1/100$ are written on the blackboard. One may delete two arbitrary numbers a and b among them and replace them by the number $a+b+ab$. After 99 such operations, only one number is left. What are all the possible values of the final number?

Q 2.16. A function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ satisfies $f(m+n) = mn + f(m) + f(n) + 1$.
Find the value of $\displaystyle\sum_{n=-24}^{24} f(n)$.

Q 2.17. The sequence (a_n) is defined as follows: $a_1 = 1,\ a_2 = 2$ and

$$a_{n+2} = \frac{2}{a_{n+1}} + a_n \text{ for } n \geq 1.$$

What is $a_{100} \cdot a_{101}$?

Q 2.18. Let $H_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$ and $T_n = \dfrac{1}{(n+1)H_n H_{n+1}}$.
Evaluate $\displaystyle\sum_{n=1}^{\infty} T_n$.

Q 2.19. Evaluate the following expression and give your answer in the simplest form:

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{\cdots+\cfrac{1}{2019}}}}} + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{\cdots+\cfrac{1}{2019}}}}}}.$$

Q 2.20. Six numbers are placed on a circle. For every number A on the circle, we have: $A = |B - C|$, where B and C follow A clockwise. The total sum of the numbers equal 1. State the numbers in cyclic order, starting with the smallest.

Q 2.21. Find the smallest positive real number c such that the following inequality holds for all non-negative reals x and y :

$$\sqrt{xy} + c|x - y| \geq \frac{x + y}{2}$$

Q 2.22. Solve the following inequality for positive x :

$$x(8\sqrt{1 - x} + \sqrt{1 + x}) \leq 11\sqrt{1 + x} - 16\sqrt{1 - x}.$$

Q 2.23. Let p be an odd prime. Let x and y with $x < y$ be positive integers that satisfy

$$2xy = (x + p)(y + p).$$

What is the sum of all possible values of x?

Q 2.24. For any positive integer n, let $s(n)$ denote the number of ways that n can be written as a sum of 1s and 2s, where the order matters.
As an example, $s(3) = 3$ as $3 = 1 + 1 + 1 = 1 + 2 = 2 + 1$.
Evaluate

$$\lim_{n \to \infty} \frac{s(n - 1)}{s(n)}.$$

3 Brief Thought

Q 3.1. Prove that the probability that an integer is prime is 0. In other words, prove that if $\pi(N)$ denotes the number of primes $\leq N$, then

$$\lim_{N\to\infty} \frac{\pi(N)}{N} = 0.$$

Q 3.2. The circumference of a circle is divided into p equal parts by the points $A_1, A_2, \cdots A_p$, where p is an odd prime number. How many different self-intersecting p-gons are there with these points as vertices if two p-gons are considered different only when neither of them can be obtained from the other by rotating the circle? (A self-intersecting polygon is a polygon some of whose sides intersect at other points besides the vertices).

Q 3.3. The set of positive integers is represented as a union of pairwise disjoint subsets, whose elements form infinite arithmetic progressions with positive differences $d_1,\ d_2,\ d_3,\ \cdots$. Is it possible that the sum

$$\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} + \cdots$$

does not exceed 0.9? Consider the cases where

(a) the total number of progressions is finite, and

(b) the number of progressions is infinite. (In this case, the condition that

$$\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} + \cdots$$

does not exceed 0.9 should be taken to mean that the sum of any finite number of terms does not exceed 0.9.)

Q 3.4. A table has m rows and n columns where m and n are positive integers greater than 1. The following permutations of its mn elements are permitted: an arbitrary permutation leaving each element in the same row (a "horizontal move") and an arbitrary permutation leaving each element in the same column (a "vertical move"). Find the number k such that any permutation of mn can be obtained by l permitted moves but there exists a permutation that cannot be achieved in less than k moves.

4 Challenge

Q 4.1. (Number Theory)
Let S be a subset of positive integers.
n belongs to S if and only if there exists a circle in the XY plane which has exactly n lattice points in its interior (excluding the boundary).
Describe the set S.

Q 4.2. (Functional Equation)
Let $f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ and $g(x) = x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m$ be two polynomials with real coefficients such that for each real number x, $f(x)$ is the square of an integer if and only if so is $g(x)$.
Prove that if $n + m > 0$, there exists a polynomial $h(x)$ with real coefficients such that $f(x) \cdot g(x) = (h(x))^2$.

Q 4.3. (Combinatorics)
Amy and Sheldon play a game in the following manner:
Amy picks a positive integer N and tells Sheldon.
Sheldon then chooses a collection of eleven (not necessarily distinct) integers $(a_i)_{i=1}^{11}$.
Now, Amy must choose a collection $(b_i)_{i=1}^{11}$ where each b_i is either $-1,\ 0,\ 1$ and at least one b_i is non-zero.
After this, the sum $S = \sum_{i=1}^{11} a_ib_i$ is computed. Amy wins the game iff S is divisible by N, otherwise Sheldon wins.
Assuming Amy and Sheldon play optimally, what is the largest value of N that Amy can pick such that she can win?

Q 4.4. (Geometry)
On a plane, a square is given, and 2019 equilateral triangles are inscribed in this square. All vertices of any of these triangles lie on the border of the square. Prove that one can find a point on the plane belonging to the borders of no less than 505 of these triangles.

Q 4.5. (Inequality)
The numbers $1, 2, \ldots, N$ are written on a board where N is a positive integer strictly greater than 1729. Sheldon performs an operation where he erases four numbers of the form $a,\ b,\ c,\ a+b+c$ and then writes $a+b,\ b+c,\ c+a$ in their place.
Prove that he can do this operation no more than $\left\lfloor \dfrac{N(N-1)}{2(2N+1)} \right\rfloor$ times.

Q 4.6. (Probability) Let $n \geq 4$ be given, and suppose that the points $P_1,\ P_2,\ \ldots,\ P_n$ are randomly chosen on a circle. Consider the convex $n-$gon whose vertices are these points. What is the probability that at least one of the vertex angles of this polygon is acute?